Lecture 4. Parametric solutions of linear systems

Def A column vector is a matrix with one column.

Note (1) We will usually refer to a column vector simply as a vector.

(2) We will rarely use the term "row vector" for a matrix with one row.

(3) We write $\mathbb{R}^n$ for the set of all column vectors with n entries.

e.g. $\begin{bmatrix} 2 \\ 0 \end{bmatrix} \in \mathbb{R}^2, \quad \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \in \mathbb{R}^3$

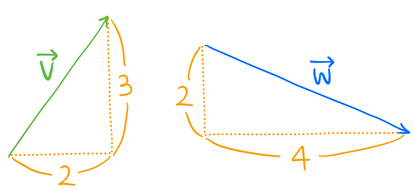
(4) To a given vector in $\mathbb{R}^n$, we can add another vector in $\mathbb{R}^n$ or multiply a constant.

e.g. $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 1+3 \\ -1+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

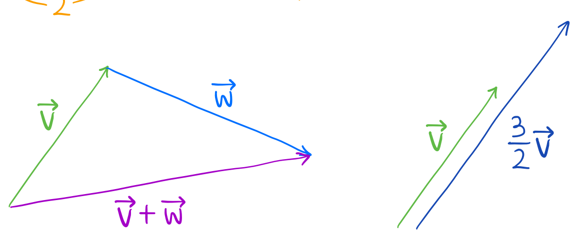
$3\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 \\ 3\cdot 1 \\ 3\cdot(-1) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}$

(5) We can visualize vectors as arrows

e.g.



$\Rightarrow \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$



Prop If a linear system in variables $x_1, x_2, \cdots, x_n$ is solvable, the general solution for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is parametrized by

$$\vec{x} = \vec{v}_0 + t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_d\vec{v}_d \quad \text{with } t_1, t_2, \cdots, t_d \in \mathbb{R}$$

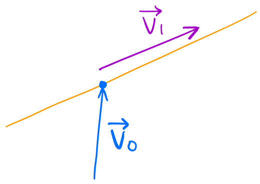
for some vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_d \in \mathbb{R}^n$.

Note (1) The parameters $t_1, t_2, \cdots, t_d$ come from free variables

$\Longrightarrow$ $d = \#$ of free variables

(2) For $d = 1$, the solution set is given by a line.

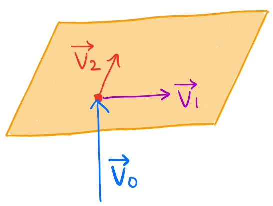
$\vec{x} = \vec{v}_0 + t_1\vec{v}_1$



free to move along the direction of $\vec{v}_1$

(3) For $d = 2$, the solution set is given by a plane.

$\vec{x} = \vec{v}_0 + t_1\vec{v}_1 + t_2\vec{v}_2$



free to move along the directions of $\vec{v}_1, \vec{v}_2$

Ex Parametrize the general solution of each linear system.

(1) $\begin{cases} x_1 + 3x_2 + x_3 = 1 \\ -4x_1 - 9x_2 + 2x_3 = -1 \\ -3x_2 - 6x_3 = -3 \end{cases}$

Sol We consider the matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - 5x_3 = -2 \\ x_2 + 2x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 + 5x_3 \\ x_2 = 1 - 2x_3 \end{cases}$$

Set $x_3 = t$ (free variable)

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2+5t \\ 1-2t \\ t \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}$$

(2) $\begin{cases} 2x_1 - 4x_2 + x_3 + 5x_4 = 8 \\ x_1 - 2x_2 + x_4 = 2 \\ 3x_1 - 6x_2 - 2x_3 - 3x_4 = -2 \end{cases}$

Sol We consider the matrix of the system.

$$\left[\begin{array}{cccc|c} 2 & -4 & 1 & 5 & 8 \\ 1 & -2 & 0 & 1 & 2 \\ 3 & -6 & -2 & -3 & -2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - 2x_2 + x_4 = 2 \\ x_3 + 3x_4 = 4 \end{cases} \Rightarrow \begin{cases} x_1 = 2 + 2x_2 - x_4 \\ x_3 = 4 - 3x_4 \end{cases}$$

Set $x_2 = s$ and $x_4 = t$ (free variables)

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2+2s-t \\ s \\ 4-3t \\ t \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}}$$

Ex For each linear system, classify its solution set as an empty set, a point, a line, or a plane.

(1) $2x+6y-4z=2$

Sol We consider the matrix of the system.

$$\left[\begin{array}{ccc|c} 2 & 6 & -4 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \end{array}\right]$$

The last column does not contain a leading 1.

Column 2 and column 3 do not contain a leading 1.

$\Rightarrow$ The system is solvable with 2 free variables

$\Rightarrow$ The solution set is $\boxed{\text{a plane}}$

Note In fact, an equation of the form $ax+by+cz=d$ represents a plane.

(2) $\begin{cases} x+y+z=2 \\ x-2y+4z=5 \end{cases}$

Sol We consider the matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -2 & 4 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

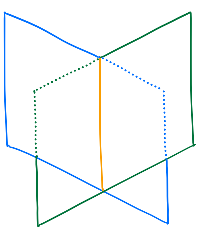
The last column does not contain a leading 1.

Column 3 does not contain a leading 1.

$\Rightarrow$ The system is solvable with 1 free variable

$\Rightarrow$ The solution set is $\boxed{\text{a line}}$

Note Geometrically, the system represents 2 planes which are not parallel.



(3) $\begin{cases} x-2y+3z=1 \\ 2x-4y+6z=3 \end{cases}$

Sol We consider the matrix of the system.

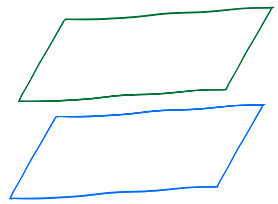
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last column contains a leading 1.

$\Rightarrow$ The system is not solvable

$\Rightarrow$ The solution set is $\boxed{\text{an empty set}}$

Note Geometrically, the system represents 2 planes which are parallel.



(4) $\begin{cases} x+3y+5z=4 \\ 3x+5y+7z=8 \\ 5x+7y+z=4 \end{cases}$

Sol We consider the matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 3 & 5 & 7 & 8 \\ 5 & 7 & 1 & 4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The last column does not contain a leading 1.

All other columns contain a leading 1.

$\Rightarrow$ The system is solvable with a unique solution

$\Rightarrow$ The solution set is $\boxed{\text{a point}}$

Note Geometrically, the system represents 3 planes which intersect at a point.